

ON SPECTRAL N -BERNOULLI MEASURES

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ABSTRACT. For $0 < \rho < 1$ and $N > 1$ an integer, let μ be the self-similar measure defined by $\mu(\cdot) = \sum_{i=0}^{N-1} \frac{1}{N} \mu(\rho^{-1}(\cdot - i))$. We prove that $L^2(\mu)$ has an exponential orthonormal basis if and only if $\rho = \frac{1}{q}$ for some $q > 0$ and N divides q . The special case is the Cantor measure with $\rho = \frac{1}{2k}$ and $N = 2$ [15], which was proved recently to be the only spectral measure among the Bernoulli convolutions with $0 < \rho < 1$ [4].

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1. Introduction

Let μ be a probability measure on \mathbb{R}^s with compact support. For a countable subset $\Lambda \subset \mathbb{R}^s$, we let $e_\Lambda = \{e_\lambda = e^{-2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda\}$. We call μ a spectral measure, and Λ a *spectrum* of μ if e_Λ is an orthogonal basis for $L^2(\mu)$. The existence and nonexistence of a spectrum for μ is a basic problem in harmonic analysis, it was initiated by Fuglede in his seminal paper [12], and has been studied extensively since then [4, 5, 7–10, 13, 15, 18–20, 22, 23, 26, 27]. Recently He, Lai and Lau [13] proved that a spectral measure μ must be of pure type, i.e., μ is absolutely continuous or singular continuous with respect to the Lebesgue measure or counting measure supported on a finite set (actually this holds more generally for *frames*). When μ is the Lebesgue measure restricted on a set K in \mathbb{R}^s , it is well-known that the spectral property is closely connected with the tiling property of K , and is known as the Fuglede problem [12, 16, 18, 27]. For continuous singular measures, the first spectral measure was given by Jorgensen and Pedersen [15]: the Cantor

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measure μ_ρ with contraction ratio $\rho = 1/2k$. There are considerable studies for such measures [5, 7, 14, 18, 22, 23, 26], and a celebrated open problem was to characterize the spectral measures $\mu_\rho, 0 < \rho < 1$ among the Bernoulli convolutions

$$\mu_\rho(\cdot) = 1/2\mu_\rho(\rho^{-1}\cdot) + 1/2\mu_\rho(\rho^{-1}\cdot - 1).$$

In [14], Hu and Lau showed that μ_ρ admits an infinite orthonormal set if and only if ρ is the n -th root of p/q where p is odd and q is even. The characterization problem was finally completed recently by Dai [4] that the above Cantor measures $\mu_{1/(2k)}$ is the only class of spectral measures among the μ_ρ .

In this paper we study the spectrality of the self-similar measures. Let $0 < \rho < 1$, $\mathcal{D} = \{0, d_1, \dots, d_{N-1}\}$ a finite set in \mathbb{R} , and $\{w_j\}_{j=0}^{N-1}$ a set of probability weights. We call μ a *self-similar measure* generated by (ρ, \mathcal{D}) and $\{w_j\}_{j=0}^{N-1}$ if μ is the unique probability measure satisfying

$$(1.1) \quad \mu(\cdot) = \frac{1}{N} \sum_{j=0}^{N-1} w_j \mu(\rho^{-1}(\cdot) - d_j).$$

We will use $\mu_{\rho,N}$ to denote the special case where $\mathcal{D} = \{0, \dots, N-1\}$ with uniform weight, i.e.,

$$(1.2) \quad \mu_{\rho,N}(\cdot) = \frac{1}{N} \sum_{j=0}^{N-1} \mu_{\rho,N}(\rho^{-1}(\cdot) - j).$$

The spectral property of such measure was first studied by Dai, He and Lai [5] as a generalization of the Bernoulli convolution in [4] ($\mathcal{D} = \{0, 1\}$). Our main result in this paper is to extend the characterization of spectral Bernoulli convolution to the class of $\mu_{\rho,N}$ in (1.2). Our motivation to extend the Bernoulli convolutions to this class of measures is due to a conjecture of Łaba and Wang, and also to answer a question on the convolution of spectral measures (see the remark in §6). We prove

Theorem 1.1. *Let $0 < \rho < 1$. Then $\mu_{\rho,N}$ is a spectral measure if and only if $\rho = \frac{1}{q}$ for some integer $q > 1$ and $N \mid q$.*

The sufficiency of the theorem follows from the same pattern as the Cantor measure in [15] by producing a Hardamard matrix, then construct the canonical spectrum (see Section 2). Ont the other hand, the proof of the necessity needs more work. We observe that for $\mu_{\rho,N}$ to be a spectral measure, ρ must be an algebraic number. we prove by elimination that each of the following cases can NOT admit a spectrum: (for $\frac{p}{q}$, we always assume they have no common factor)

- (i) $\rho = (\frac{p}{q})^{1/r}$ for some $r > 1$ (it is an irrational), (Proposition 3.1);
- (ii) $\rho \neq (\frac{p}{q})^{1/r}$ for any $r > 1$ and is an irrational (Proposition 3.4);
- (iii) $\rho = \frac{p}{q}$ and $1 \leq \gcd(N, q) < N$ (Proposition 5.1);
- (iv) $\rho = \frac{p}{q}$, $p > 1$ and $N \mid q$ (Proposition 5.2).

Let $\widehat{\mu}_{\rho,N}$ be the Fourier transform of $\mu_{\rho,N}$, and $\mathcal{Z}(\widehat{\mu}_{\rho,N})$ the zeros of $\widehat{\mu}_{\rho,N}$. The proof is based on the criteria in Theorem 2.1 and Lemma 2.2, and the technique is to make use of some explicit expressions of $\mathcal{Z}(\widehat{\mu}_{\rho,N})$, and that $\Lambda - \Lambda \subset \mathcal{Z}(\widehat{\mu}_{\rho,N})$ for any exponential orthogonal set Λ .

The most subtle part of the proof is (iv). As is known, there is certain canonical q -adic expansion of λ in a spectrum Λ (see (2.1)), and there are also others. In [7], Dutkay *et al* treated the 4-adic expansions as in a symbolic space Ω_2^* , and consider certain maps on Ω_2^* to \mathbb{Z}^+ to preserve the maximal orthogonality property. This idea was refined and investigated by Dai, He and Lai [5] by replacing the q -adic expansion on \mathbb{Z} with digits in $C = \{-1, 0, \dots, q-2\}$. Let $\iota : \Omega_N^* \rightarrow C$ be a *selection map* as defined in Definition 4.3 (it was called a *maximal map* in [5]), and let $\iota^*(\mathbf{i}) = \sum_{n=1}^{\infty} \iota(\mathbf{i}0^\infty|_n)q^{n-1}$. The importance of the selection map is in the following theorem (Theorem 4.5), which also has independent interest.

Theorem 1.2. *Suppose $\rho = p/q$ and $N \mid q$. Then $\Lambda \subset \mathcal{Z}(\widehat{\mu}_{\rho,N})$ defines a maximal exponential orthogonal subset in $L^2(\mu_{\rho,N})$ if and only if there exist $m_0 \geq 1$ and a selection map ι such that $\Lambda = \rho^{-m_0}N^{-1}(\iota^*(\Omega_N^*))$.*

We organize the paper as follows. In Section 2, we set up the notations, the basic criteria of spectrum, and the element properties of the zero set $\mathcal{Z}(\mu_{\rho,N})$. We settle cases (i), (ii) in Section 3. For the case $\rho = p/q$, in Section 4 we give a detail study of the maximality of Λ such the $\Lambda - \Lambda \subset \mathcal{Z}(\mu_{\rho,N})$, which is used in Section 5 to consider cases (iii) and (iv). In Section 6, we give some remarks of the spectral measures and the remaining questions.

2. Preliminaries

We assume that μ is a probability measure with compact support. The Fourier transformation of μ is define as usual,

$$\widehat{\mu}(\xi) = \int e^{-2\pi i \xi x} d\mu(x).$$

Let $\mathcal{Z}(\widehat{\mu}) := \{\xi : \widehat{\mu}(\xi) = 0\}$ be the set of zeros of $\widehat{\mu}$. We denote the complex exponential function $e^{-2\pi i \lambda(\cdot)}$ by e_λ . Note that $\{e_\lambda : \lambda \in \Lambda\}$ is an orthogonal set in $L^2(\mu)$ if and only if $\widehat{\mu}(\lambda_i - \lambda_j) = 0$ for any $\lambda_i \neq \lambda_j \in \Lambda$; Λ is called a *spectrum* of μ if $\{e_\lambda\}_{\lambda \in \Lambda}$ is an orthonormal basis for $L^2(\mu)$. For $\xi \in \mathbb{R}$, we let

$$Q(\xi) = \sum_{\lambda \in \Lambda} |\widehat{\mu}(\xi + \lambda)|^2.$$

The following theorem is a basic criterion for the spectrality of μ [15].

Theorem 2.1. *Let μ be a probability measure with compact support, and let $\Lambda \subset \mathbb{R}$ be a countable subset. Then*

- (i) $\{e_\lambda\}_{\lambda \in \Lambda}$ is an orthonormal set of $L^2(\mu)$ if and only if $Q(\xi) \leq 1$ for $\xi \in \mathbb{R}$; and
- (ii) it is an orthonormal basis if and only if $Q(\xi) \equiv 1$ for $\xi \in \mathbb{R}$.

Throughout the paper, we use the notation Λ to denote a subset such that $0 \in \Lambda$ and $\Lambda \setminus \{0\} \subset \mathcal{Z}(\widehat{\mu})$. We say that Λ is a *bi-zero set* of μ if $(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\widehat{\mu})$, and call it a *maximal bi-zero set* if it is maximal in $\mathcal{Z}(\widehat{\mu})$ to have the set difference property. Clearly that Λ is a bi-zero set is equivalent to $\{e_\lambda : \lambda \in \Lambda\}$ is an orthogonal subset of $L^2(\mu)$. An exponential orthonormal basis corresponds to a maximal bi-zero set, but the converse is not true. In fact we will give a characterize of the maximal bi-zero sets of $\mu_{\rho,N}$ for the case $\rho = \frac{p}{q}$ and $N \mid q$ in Section 4, and establish the spectrality through Theorem 2.1(ii) in Section 5.

As a simple consequence of Theorem 2.1, we have the following useful lemma.

Lemma 2.2. *Let $\mu = \mu_0 * \mu_1$ be the convolution of two probability measures μ_i , $i = 0, 1$, and they are not Dirac measures. Suppose that Λ is a bi-zero set of μ_0 , then Λ is also a bi-zero of μ , but cannot be a spectrum of μ .*

Proof. Note that μ_i is not an Dirac measure is equivalent to $|\widehat{\mu}_i(\xi)| \not\equiv 1$. Since $\widehat{\mu}_0(0) = 1$, there exists ξ_0 such that $|\widehat{\mu}_0(\xi_0)| \neq 0$ and $|\widehat{\mu}_1(\xi_0)| < 1$. Hence by Theorem 2.1(i),

$$\mathcal{Q}(\xi_0) = \sum_{\lambda \in \Lambda} |\widehat{\mu}(\xi_0 + \lambda)|^2 = \sum_{\lambda \in \Lambda} |\widehat{\mu}_0(\xi_0 + \lambda)|^2 |\widehat{\mu}_1(\xi_0 + \lambda)|^2 < \sum_{\lambda \in \Lambda} |\widehat{\mu}_0(\xi_0 + \lambda)|^2 \leq 1.$$

The result follows by Theorem 2.1(i) and (ii). \square

Now we consider the self-similar measure $\mu_{\rho,N}$ in Theorem 1.1. It was proved in [5] that if $\rho = 1/q$ and $N \mid q$, then $\mu_{\rho,N}$ is a spectral measure. The proof is quite simple. In fact as $N \mid q$, we write $q = Nr$. If $r = 1$, then μ is just the Lebesgue measure on the unit interval, and the result is trivial. If $r > 1$, observe that for $\mathcal{D} = \{0, \dots, N-1\}$ and $\Gamma = r\{0, \dots, N-1\}$, the matrix

$$H := [e^{2\pi i \frac{ik}{q}}]_{i \in \mathcal{D}, k \in \Gamma} = [e^{2\pi i \frac{ij}{N}}]_{0 \leq i, j \leq N}$$

is a Hadamard matrix (i.e., $HH^* = NI$). This shows that $(q^{-1}\mathcal{D}, \Gamma)$ is a *compatible pair*, hence $\mu_{1/q,N}$ is a spectral measure [18], and the canonical spectrum is given by

$$(2.1) \quad \Lambda = \left\{ \sum_{j=0}^k a_j q^j : a_j \in \Gamma, k \geq 0 \right\}$$

(note that the spectrum is not unique). Our main task is to prove the converse. The strategy is to eliminate all the possible cases so that the only admissible case is $\rho = 1/q$ with $N \mid q$.

Recall that the Fourier transform of $\mu_{\rho,N}$ has the following expression

$$\widehat{\mu}_{\rho,N}(\xi) = M_N(\rho\xi) \widehat{\mu}_{\rho,N}(\rho\xi) = \prod_{k=1}^{\infty} M_N(\rho^k \xi)$$

where $M_N(\xi) = \frac{1}{N} \sum_{j=0}^{N-1} e^{-2\pi i j \xi}$ is the *mask polynomial* of \mathcal{D} . It is clear that $|M_N(\xi)| = \left| \frac{\sin N\pi\xi}{N \sin \pi\xi} \right|$, and the zeros of $M_N(\xi)$ is a/N , $a \in \mathbb{Z} \setminus \{0\}$, $N \nmid a$. Let

$$(2.2) \quad \mathcal{Z}(M_N) = \left\{ \frac{a}{N} : a \in \mathbb{Z} \setminus \{0\}, N \nmid a \right\} = \left\{ \frac{a}{N} : a \in \mathbb{Z} \setminus N\mathbb{Z} \right\}.$$

It follows from the infinite product expression of $\widehat{\mu}_{\rho,N}$ that

$$(2.3) \quad \mathcal{Z}(\widehat{\mu}_{\rho,N}) = \{\rho^{-k} \frac{a}{N} : k \geq 1, a \in \mathbb{Z} \setminus N\mathbb{Z}\}.$$

For distinct $\lambda_1, \lambda_2 \in \Lambda \setminus \{0\}$, (2.3) and the bi-zero property of Λ imply that

$$\rho^{-k_1} \frac{a_1}{N} - \rho^{-k_2} \frac{a_2}{N} = \rho^{-k} \frac{a}{N}.$$

Hence ρ is an algebraic number. Recall that an *algebraic number* is a root of an integer equation of the form $c_0 x^n + c_1 x^{n-1} + \dots + c_n \in \mathbb{Z}[x]$, and it is called an *algebraic integer* if $c_0 = 1$.

3. Spectrality for irrational contraction

For any integer $r \geq 1$, let

$$\mathbb{Q}^{1/r} = \{\rho = u^{1/r} : 0 < u < 1 \text{ is a rational}\}.$$

We make the convention that the above r is the smallest integer for $\rho = u^{1/r}$ (for example: $\rho = (\frac{4}{9})^{1/4} = (\frac{2}{3})^{1/2}$, we will take $r = 2$). Hence for $\rho \in \mathbb{Q}^{1/r}$, $r > 1$, then ρ is an irrational.

Proposition 3.1. *Let $\rho \in \mathbb{Q}^{1/r}$, $r > 1$, then $\mu_{\rho,N}$ is not a spectral measure.*

Proof. Let $\rho = u^{1/r}$ where $0 < u < 1$ is a rational. We write

$$\widehat{\mu}_{\rho,N}(\xi) = \prod_{k=1}^{\infty} M_N(\rho^k \xi) = \prod_{k=0}^{\infty} \prod_{i=1}^r M_N(u^k \rho^i \xi).$$

Define the probability measures $\mu_i(\cdot) = \mu_{u,N}(u\rho^{-i}\cdot)$, $1 \leq i \leq r$. Then

$$\widehat{\mu}_i(\xi) = \prod_{k=0}^{\infty} M_N(u^k \rho^i \xi)$$

for $1 \leq i \leq r$. Then μ_{ρ} is the convolution of μ_i , $i = 1, 2, \dots, r$. Let Λ be a bi-zero set of $\mu_{\rho,N}$. We claim that Λ is also a bi-zero set of μ_i for some i . Indeed, let $\lambda_j = \rho^{-k_j r - i_j} a_j / N$, $1 \leq i_1, i_2 \leq r, j = 1, 2$, be any two distinct elements in Λ . The bi-zero property of Λ for μ implies that

$$\rho^{-k_1 r - i_1} a_1 / N - \rho^{-k_2 r - i_2} a_2 / N = \rho^{-kr - i} a / N$$

Without loss of generality assume $k_1, k_2 \geq k$, then we have $u^{(k_1 - k)} \rho^{i_1 - i} a_1 - u^{(k_2 - k)} \rho^{i_2 - i} a_2 = a$. This implies $i_1 = i_2 = i$ because the minimal polynomial of ρ is $x^r - u$. Hence Λ is a bi-zero set of μ_i , and by Lemma 2.2, Λ cannot be a spectrum of μ . \square

Next we consider $\rho \notin \mathbb{Q}^{1/r}$, $r > 1$. We need two lemmas.

Lemma 3.2. *Suppose Λ is an infinity bi-zero set of $\mu_{\rho,N}$ with $0 \in \Lambda$. Then $\rho \notin \mathbb{Q}^{1/r}$ for all $r \geq 1$ implies that ρ is an algebraic integer.*

Proof. Since $\Lambda \setminus \{0\} \subset \mathcal{Z}(\widehat{\mu}_{\rho,N})$, we denote $\Lambda = \{\lambda_k\}_{k=0}^\infty$ so that $\lambda_0 = 0$ and $\lambda_k = \rho^{-n_k} \frac{a_k}{N}$, where $N \nmid a_k$ for $k \geq 1$. We can assume that $n_k \leq n_{k+1}$ for $k \geq 1$. Fix $\ell \geq 1$. For any integer $G > 0$ and $k > \ell$, by the bi-zero property of Λ , we have

$$\lambda_k - \lambda_\ell = \rho^{-n_{k,\ell}} \frac{a_{k,\ell}}{N}, \quad a_{k,\ell} \in \mathbb{Z} \setminus N\mathbb{Z}.$$

We claim $\#\{k : n_{k,\ell} \leq G\} \leq (N-1)G$. Otherwise, by the pigeon hole principle, there exist k_1, k_2 such that $n_{k_1,\ell} = n_{k_2,\ell} \leq G$ and $N \mid (a_{k_1,\ell} - a_{k_2,\ell})$. Then, by the definition of $\mathcal{Z}(\widehat{\mu}_{\rho,N})$ and $\rho \notin \mathbb{Q}^{1/r}$ for all $r \geq 1$, we have

$$\lambda_{k_1} - \lambda_{k_2} = \lambda_{k_1} - \lambda_\ell + \lambda_\ell - \lambda_{k_2} = \rho^{-n_{k_1,\ell}} \frac{a_{k_1,\ell} - a_{k_2,\ell}}{N} \notin \mathcal{Z}(\widehat{\mu}_{\rho,N}).$$

Hence the claim follows. Taking any $k > \ell$ such that $n_{k,\ell} > n_\ell$, we conclude from

$$\rho^{-n_k} \frac{a_k}{N} - \rho^{-n_\ell} \frac{a_\ell}{N} = \rho^{-n_{k,\ell}} \frac{a_{k,\ell}}{N}$$

that there exists a polynomial $p(x) = a_\ell x^s + b x^t + c$ with $s > t$ and $p(\rho) = 0$. Let $\varphi(x) = c_0 x^m + c_1 x^{m-1} + \cdots + c_m \in \mathbb{Z}[x]$ be the minimal polynomial of ρ . This implies that $\varphi(x) \mid p(x)$, and thus $c_0 \mid a_\ell$. Let ℓ run through all $\lambda_\ell \in \Lambda$. Then

$$(3.1) \quad \frac{1}{c_0} \Lambda \setminus \{0\} \subseteq \mathcal{Z}(\widehat{\mu}_{\rho,N}).$$

To show that $\frac{1}{c_0} \Lambda$ is a bi-zero set of $\mu_{\rho,N}$ we need to prove that

$$(3.2) \quad \frac{1}{c_0} (\Lambda - \Lambda) \setminus \{0\} \subseteq \mathcal{Z}(\widehat{\mu}_{\rho,N}).$$

For any $\lambda_{k_1} \neq \lambda_{k_2} \in \Lambda$, by the claim there exists k such that $\min\{n_{k,k_1}, n_{k,k_2}\} > n_{k_1,k_2}$, thus

$$\rho^{-n_{k_1,k_2}} \frac{a_{k_1,k_2}}{N} = \lambda_{k_1} - \lambda_{k_2} = (\lambda_{k_1} - \lambda_k) - (\lambda_{k_2} - \lambda_k) = \rho^{-n_{k,k_2}} \frac{a_{k,k_2}}{N} - \rho^{-n_{k,k_1}} \frac{a_{k,k_1}}{N}.$$

Similar to the above, we have $c_0 \mid a_{k_1,k_2}$. Then (3.2) holds.

By repeating the same argument, we see that $\frac{1}{c_0^k} \Lambda$ is also a bi-zero set of $\mu_{\rho,N}$ for any $k \geq 1$. This force $c_0 = 1$. \square

For any $x \in \mathbb{R}$, let $\|x\| = |\langle x \rangle|$, where $\langle x \rangle$ is the unique number such that $\langle x \rangle \in (-1/2, 1/2]$ and $x - \langle x \rangle \in \mathbb{Z}$. Clearly $\|x\|$ is the distance from x to \mathbb{Z} .

Lemma 3.3. *Let ρ be a root of $x^m + c_1 x^{m-1} + \cdots + c_m \in \mathbb{Z}[x]$. Then for any $a \in \mathbb{Z} \setminus N\mathbb{Z}$,*

$$(3.3) \quad \max_{1 \leq n \leq m} \|\rho^{-n} \frac{a}{N}\| \geq \left(N \sum_{n=1}^m |c_n| \right)^{-1} := \alpha > 0.$$

Proof. Denote $\rho^{-n} \frac{a}{N} = \langle \rho^{-n} \frac{a}{N} \rangle + k_n$, $1 \leq n \leq m$. Then

$$(3.4) \quad \frac{a}{N} + \sum_{n=1}^m c_n \langle \rho^{-n} \frac{a}{N} \rangle + \sum_{n=1}^m c_n k_n = 0.$$

If $|\langle \rho^{-n} \frac{a}{N} \rangle| < \alpha$ for $1 \leq n \leq m$, then $|\sum_{n=1}^m c_n \langle \rho^{-n} \frac{a}{N} \rangle| < \frac{1}{N}$. This contradicts (3.4) as $a \in \mathbb{Z} \setminus N\mathbb{Z}$. Hence the result follows. \square

Proposition 3.4. *Let ρ be an irrational and $\rho \notin \mathbb{Q}^{1/r}$ for any $r > 1$. Then $\mu_{\rho,N}$ is not a spectral measure.*

Proof. Suppose on the contrary that $\mu_{\rho,N}$ is a spectral measure. Then, by Lemma 3.2, ρ is an algebra integer, and $\phi(x) = x^m + c_1 x^{m-1} + \cdots + c_m \in \mathbb{Z}[x]$ is the minimal polynomial of ρ .

Let Λ be a spectrum of $\mu_{\rho,N}$ with $0 \in \Lambda$. Denote $\Lambda_k = \Lambda \cap \{\rho^{-k} \frac{a}{N} : a \in \mathbb{Z} \setminus N\mathbb{Z}\}$ for $k \geq 1$. Then $\#\Lambda_k \leq N-1$ for $k \geq 1$ (by the proof of Lemma 3.2). Let $M_N(\xi)$ be the mask polynomial and let $G(\xi) = \sum_{i=1}^{N-1} |M_N(\xi + \frac{i}{N})|^2$. Then by applying Theorem 2.1 to the point mass measure $\frac{1}{N}\delta_{\{0, \dots, N-1\}}$, we have

$$G(\xi) + |M_N(\xi)|^2 = \sum_{i=0}^{N-1} |M_N(\xi + \frac{i}{N})|^2 = 1,$$

and hence $G(0) = 0$. Observing that $G(z)$ is an entire function, then there exists an entire function $H(z)$ and integer $t > 0$ such that $G(z) = z^t H(z)$ and $H(0) \neq 0$. To prove that $Q(\xi) = |\widehat{\mu}_{\rho,N}(\xi)|^2 + \sum_{k=1}^{\infty} \sum_{\lambda \in \Lambda_k} |\widehat{\mu}_{\rho,N}(\xi + \lambda)|^2 \neq 1$, we first observe that for any ξ ,

$$\begin{aligned} (3.5) \quad \sum_{\lambda \in \Lambda_k} |\widehat{\mu}_{\rho,N}(\xi + \lambda)|^2 &= \sum_{\lambda \in \Lambda_k} \prod_{j=1}^k |M_N(\rho^j(\xi + \lambda))|^2 \cdot |\widehat{\mu}_{\rho,N}(\rho^k(\xi + \lambda))|^2 \\ &\leq \sum_{\lambda \in \Lambda_k} |M_N(\rho^k(\xi + \lambda))|^2 \\ &\leq G(\rho^k \xi). \end{aligned}$$

(The last inequality follow from $\lambda \in \Lambda_k, \rho^k \lambda = \frac{a}{N} \neq 0, a \nmid N$). Let m and $\alpha (< 1/2)$ be defined as in Lemma 3.3, and let $\beta = \min\{1 - |M_N(x)|^2 : \alpha/2 \leq |x| \leq 1 - \alpha/2\}$. Then obviously $\beta > 0$. Note that for each $k > m$ and $\lambda \in \Lambda_k$,

$$\rho^j \lambda = \rho^{-(k-j)} \frac{a}{N}, \quad j = 1, 2, \dots, k-1.$$

Hence for $0 \leq \xi \leq \alpha/2, k > m$, by Lemma 3.3, there exist $k-m \leq \ell_\lambda \leq k-1$ such that $\|\rho^{\ell_\lambda}(\xi + \lambda)\|^2 \geq \alpha/2$. Hence from (3.5), we have

$$\begin{aligned} \sum_{\lambda \in \Lambda_k} |\widehat{\mu}_{\rho,N}(\xi + \lambda)|^2 &\leq \sum_{\lambda \in \Lambda_k} |M_N(\rho^{\ell_\lambda}(\xi + \lambda))|^2 \cdot |M_N(\rho^k(\xi + \lambda))|^2 \\ &\leq (1 - \beta) \sum_{\lambda \in \Lambda_k} |M_N(\rho^k(\xi + \lambda))|^2 \\ &\leq (1 - \beta) G(\rho^k \xi). \end{aligned}$$

Note that $\Lambda \setminus \{0\} = \cup_{k \in \mathbb{N}} \Lambda_k$, and $\Lambda_{k_1} \cap \Lambda_{k_2} = \emptyset$ when $k_1 \neq k_2$ since $\lambda \notin \mathbb{Q}^{\frac{1}{r}}$ for all $r \in \mathbb{N}$. Hence, by (3.5) and (3.6),

$$\begin{aligned}
Q(\xi) &= \sum_{\lambda \in \Lambda} |\widehat{\mu}_{\rho,N}(\xi + \lambda)|^2 \\
(3.6) \quad &= |\widehat{\mu}_{\rho,N}(\xi)|^2 + \sum_{k=1}^{\infty} \sum_{\lambda \in \Lambda_k} |\widehat{\mu}_{\rho,N}(\xi + \lambda)|^2 \\
&\leq |\widehat{\mu}_{\rho,N}(\xi)|^2 + \sum_{k=1}^m G(\rho^k \xi) + (1 - \beta) \sum_{k>m} G(\rho^k \xi).
\end{aligned}$$

On the other hand, recall that $G(z) = z^t H(z)$ and $H(0) = 0$, then $0 < C_1 \leq |H(z)| \leq C_2$ if $|z| \leq \eta \leq \alpha/2$ for some small η . Therefore for $0 \leq \xi \leq \eta$,

$$(3.7) \quad \frac{C_1 \rho^{mt}}{1 - \rho^t} \xi^t \leq \sum_{k=m}^{\infty} G(\rho^k \xi) \leq \frac{C_2 \rho^{mt}}{1 - \rho^t} \xi^t,$$

and

$$\begin{aligned}
|\widehat{\mu}_{\rho,N}(\xi)|^2 &= \prod_{k=1}^{\infty} |M_N(\rho^k \xi)|^2 = \prod_{k=1}^{\infty} (1 - G(\rho^k \xi)) \\
(3.8) \quad &\leq e^{-\sum_{k=1}^{\infty} G(\rho^k \xi)} \leq 1 - \sum_{k=1}^{\infty} G(\rho^k \xi) + o\left(\sum_{k=1}^{\infty} G(\rho^k \xi)\right),
\end{aligned}$$

where $o(\xi)$ satisfies that $\lim_{\xi \rightarrow 0} o(\xi)/\xi = 0$. Hence, by (3.6) and (3.8), we have

$$(3.9) \quad Q(\xi) \leq 1 - \beta \sum_{k=m}^{\infty} G(\rho^k \xi) + o\left(\sum_{k=1}^{\infty} G(\rho^k \xi)\right).$$

By (3.7) this implies $Q(\xi) < 1$ for $\xi > 0$ small enough. That Λ cannot be a spectrum follows by Theorem 2.1. \square

In view of Propositions 3.1 and 3.4, we have to prove that $\mu_{\rho,N}$ cannot be a spectral measure in the remaining cases (iii) and (iv) in §1 for $\rho = p/q$. These will be proved in the remaining sections.

4. Structure of bi-zero sets for rational contraction

In this section we will consider $\rho = p/q$, we assume p, q are co-primes throughout. Let $\Lambda = \{\lambda_k\}_{k=0}^{\infty} \subseteq \mathcal{Z}(\widehat{\mu}_{\rho,N})$ (with $\lambda_0 = 0$) be a bi-zero set of $\mu_{\rho,N}$. Then by (2.3),

$$(4.1) \quad \lambda_k = \left(\frac{q}{p}\right)^{n_k} \frac{a_k}{N} \quad \text{with } a_k \in \mathbb{Z} \setminus N\mathbb{Z}, \quad k \geq 1.$$

In the following, we will give another expression of the λ_k which is more convenient to use here.

Lemma 4.1. *Let Λ be a bi-zero set of $\mu_{\rho,N}$ with $\rho = \frac{p}{q}$. Then there exists $m_0 > 0$ such that each $\lambda_k \in \Lambda \setminus \{0\}$ admits an expression*

$$(4.2) \quad \lambda_k = p^{-m_0} q^{m_k} \frac{c_k}{N} \quad \text{with } c_k \in \mathbb{Z} \setminus q\mathbb{Z} \text{ and } m_k \geq m_0$$

(note that N can be a factor of c_k). Moreover, if $N \mid q$, then we can write

$$\lambda_k = p^{-m_0} q^{m_k} \frac{c_k}{N} \quad \text{with } c_k \in \mathbb{Z} \setminus N\mathbb{Z} \text{ and } m_k \geq m_0.$$

Proof. For the expression of λ_k in (4.1), we let $a_k = a'_k q^{l_k}$ so that $q \nmid a'_k$. Then we can write λ_k as

$$(4.3) \quad \lambda_k = \left(\frac{q}{p}\right)^{n_k+l_k} \frac{a'_k p^{l_k}}{N} := \left(\frac{q}{p}\right)^{m_k} \frac{b_k}{N},$$

where q is not a factor of b_k for $k \geq 1$. Let $m_0 \geq 1$ be the smallest among all such m_k , and denote the corresponding $\lambda_i \in \Lambda$ by $\left(\frac{q}{p}\right)^{m_0} \frac{b_i}{N}$. Then by the bi-zero property, for any $m_k > m_0$,

$$\left(\frac{q}{p}\right)^{m_0} \frac{b_i}{N} - \left(\frac{q}{p}\right)^{m_k} \frac{b_k}{N} = \left(\frac{q}{p}\right)^m \frac{b}{N}.$$

It is easy to see that $m = m_0$, and then $p^{m_k-m_0}$ is a factor of b_k . It follows from this that we can rewrite λ_k as

$$\lambda_k = p^{-m_0} q^{m_k} \frac{c_k}{N},$$

where $q \nmid c_k$ for $k \geq 1$.

The second assertion follows by observing that the l_k in (4.3) is zero (as $q \nmid a_k$ follows by $N \mid q$ and $N \nmid a_k$). Hence the above $c_k = a_k / p^{m_k-m_0}$ is not divisible by N by (4.1). \square

Corollary 4.2. *Let Λ be a bi-zero set of $\mu_{\rho,N}$ and let $N \mid q$ and $\rho = \frac{p}{q}$. Denote $\mathcal{Q} = \{q^m a : a \in \mathbb{Z} \setminus N\mathbb{Z}, m \geq 0\}$. Then*

$$(4.4) \quad (\Lambda - \Lambda) \setminus \{0\} \subseteq \frac{1}{\rho^{m_0} N} \mathcal{Q} \subset \mathcal{Z}(\widehat{\mu}_{\rho,N}(\xi)),$$

where m_0 is as in Lemma 4.1.

Proof. It suffices to show that

$$(\Lambda - \Lambda) \setminus \{0\} \subseteq \frac{1}{\rho^{m_0} N} \left\{ q^m a : m \geq m_0, a \in \mathbb{Z} \setminus N\mathbb{Z} \right\}.$$

If $m_k > m_l$ for $k \neq l$, we have $\lambda_k - \lambda_l = p^{-m_0} q^{m_l} \frac{q^{m_k-m_l} c_k - c_l}{N} \in \frac{1}{\rho^{m_0} N} \mathcal{Q}$ because $N \nmid c_l$. If $m_k = m_l$ for $k \neq l$, then by Lemma 4.1,

$$\lambda_k - \lambda_l = p^{-m_0} q^{m_k} (c_k - c_l) / N = p^{-m_0} q^{m_k+\alpha} c / N$$

where $q \nmid c$. By the bi-zero property in (4.1), we have

$$p^{-m_0} q^{m_k+\alpha} c / N = \lambda_k - \lambda_l = \left(\frac{q}{p}\right)^n a / N$$

where $N \nmid a$. Then $q^{m_k+\alpha-n}c = p^{m_0-n}a$, which implies that $m_k + \alpha = n$, and thus $a = cp^{\alpha+m_k-m_0}$. Hence $N \nmid c$ and the claim follows. \square

It is well-known that every positive integer has a unique q -adic expansion. In order to do this for all integers in \mathbb{Z} , we use the q -adic expansion on the set $C = \{-1, 0, \dots, q-2\}$. In the following, we will establish a relation of the λ_k in the bi-zero set Λ with such expansion. We characterize the maximal bi-zero set by certain tree-structure. We need the addition condition that $N \mid q$, and a special selection map to be defined in the following.

Let $\Omega_N = \{0, \dots, N-1\}$ and let $\Omega_N^* = \bigcup_{k=0}^{\infty} \Omega_N^k$ be the set of finite words (by convention $\Omega_N^0 = \{\emptyset\}$). We use $\mathbf{i} = i_1 \cdots i_k$ to denote an element in Ω_N^k , and $|\mathbf{i}| = k$ is the length. For any $\mathbf{i}, \mathbf{j} \in \Omega_N^*$, \mathbf{ij} is their natural conjunction. In particular, $\emptyset\mathbf{i} = \mathbf{i}$, $\mathbf{i}0^\infty = \mathbf{i}00 \cdots$ and $0^k = 0 \cdots 0 \in \Omega_N^k$.

Definition 4.3. Suppose N, q are positive integers and $N \mid q$. We call a map $\iota : \Omega_N^* \rightarrow \{-1, 0, \dots, q-2\}$ a selection mapping if

- (i) $\iota(\emptyset) = \iota(0^n) = 0$ for all $n \geq 1$;
- (ii) for any $\mathbf{i} = i_1 \cdots i_k \in \Omega_N^k$, $\iota(\mathbf{i}) \in (i_k + N\mathbb{Z}) \cap C$, where $C = \{-1, 0, 1, \dots, q-2\}$;
- (iii) for any $\mathbf{i} \in \Omega_N^*$, there exists $\mathbf{j} \in \Omega_N^*$ such that ι vanishes eventually on $\mathbf{ij}0^\infty$, i.e., $\iota(\mathbf{ij}0^k) = 0$ for sufficient large k .

Note that $C \equiv \Omega_N \oplus N\{0, \dots, r-1\} \pmod{q}$ where $q = rN$, and ι is a selection map on each level k . More explicitly, (ii) means

$$(4.5) \quad \iota(\mathbf{i}) = \begin{cases} i_k + Nt, & \text{if } 0 \leq i_k \leq N-2, \\ i_k + Nt', & \text{if } i_k = N-1, \end{cases}$$

where $t \in \{0, \dots, r-1\}$ and $t' \in \{-1, 0, \dots, r-2\}$.

Next we let

$$\Omega_N^\iota = \{ \mathbf{i} = i_1 \cdots i_k \in \Omega_N^* : i_k \neq 0, \iota(\mathbf{i}0^n) = 0 \text{ for sufficient large } n \} \cup \{\emptyset\}$$

and for any $\mathbf{i} \in \Omega_N^\iota$ we define

$$\iota^*(\mathbf{i}) = \sum_{n=1}^{\infty} \iota(\mathbf{i}0^\infty|_n) q^{n-1},$$

Here we regard $\mathbf{i}0^\infty = \mathbf{i}00 \cdots$, and $\mathbf{i}0^\infty|_n$ denotes the word of the first n entries. Clearly $\iota^*(\emptyset) = 0$.

Let $Q = \{q^m a : a \in \mathbb{Z} \setminus N\mathbb{Z}, m \geq 0\}$ be as in Corollary 4.2, a subset $L \setminus \{0\} \subset Q$ is called a D -set of Q if $0 \in L$ and $L - L \subset Q \cup \{0\}$ (D for difference), and call it a *maximal D-set* if for any $n \in Q \setminus L$, $L \cup \{n\}$ is not a D -set. The main idea of the proof of the following theorem is in [5] (and the selection map is called a maximal map there)(see also [7]). We provide a simplified proof here for completeness.

Proposition 4.4. *Suppose $N \mid q$. Then $L \subset Q := \{q^m a : m \geq 0, a \in \mathbb{Z} \setminus N\mathbb{Z}\}$ is a maximal D-set of Q if and only if $L = \iota^*(\Omega_N^\iota)$ for some selection map ι .*

Proof. We first prove the sufficiency. For a selection map ι , it is direct to check that $L = \iota^*(\Sigma_N^\iota)$ is a D-set of $Q \subseteq \mathbb{Z}$ by the definition of ι . We need only show that L is maximal in Q . Suppose otherwise, there exists $n \notin L$ and $L \cup \{n\}$ is a D-set. We can express n uniquely as

$$(4.6) \quad n = a_0 + a_1 q + \cdots + a_\ell q^\ell, \quad a_i \in C = \{-1, 0, 1, \dots, q-2\}.$$

We claim that $a_0 = \iota(i_1)$ for some $i_1 \in \Omega_N$. If otherwise, let $j \in \Omega_N = \{0, \dots, N-1\}$ such that $a_0 \in j + N\mathbb{Z}$. In view of property (ii) of ι (or (4.5)), $N \mid (a_0 - \iota(j))$. By property (iii) of ι , there exists $\mathbf{i} = i_1 \cdots i_k \in \Omega_N^\iota$ with $i_1 = j$. Then

$$n - \iota^*(\mathbf{i}) = a_0 - \iota(j) + qb,$$

where b is an integer. Hence $n - \iota^*(\mathbf{i}) \notin Q$ (as it has a factor N , and not a factor of q). This contradicts that $L \cup \{n\}$ is a D-set of Q , and the claim follows.

Similarly, by considering $n - \iota(i_1) = n - a_0$ in (4.6), we can show that $a_1 = \iota(i_1 i_2)$ for some $0 \leq i_2 < N-1$, and so on. After finitely many steps, we have $n = \iota^*(\mathbf{i})$ for some $\mathbf{i} \in \Omega_N^\iota$, which contradicts $n \notin L$, and the sufficiency follows.

Conversely, suppose that L is a maximal D-set of Q . Denote $L = \{\lambda_k\}_{k=0}^\infty$ with $\lambda_0 = 0$. Then λ_k can be expressed by

$$\lambda_k = a_{k,0} + a_{k,1}q + \cdots + a_{k,l_k}q^{l_k} = \sum_{n=0}^\infty a_{k,n}q^n,$$

where $-1 \leq a_{k,n} \leq q-2$ for $0 \leq n \leq l_k$ and $a_{k,n} = 0$ for $n > l_k$. Note that all $a_{0,n}$ are zero. We first consider $\{a_{k,0} : k \geq 0\}$, the first coefficients of the λ_k 's. As $a_{k,0}$ can be written uniquely as $i_k + N\alpha_k \in C = \{-1, 0, \dots, q-2\}$ for some $i_k \in \Omega_N = \{0, \dots, N-1\}$. We claim that

$$(4.7) \quad \{a_{k,0} : k \geq 0\} = \{i + N\alpha_i : i \in \Omega_N\} \subseteq C.$$

(Here α_i depends only on i , but not on k , hence the set have N elements.) Indeed if $\{a_{k,0} : k \geq 0\} \not\subseteq \{i + N\alpha_i : i \in \Omega_N\}$, then there exist k_1 and k_2 such that $N \mid (a_{k_1,0} - a_{k_2,0})$. Hence

$$\lambda_{k_1} - \lambda_{k_2} = a_{k_1,0} - a_{k_2,0} + qb \notin Q,$$

(same reasoning as the above) which contradicts that L is a D-set in Q . If $\{a_{k,0} : k \geq 0\} \subseteq \{i + N\alpha_i : i \in \Omega_N\}$, then there exists $0 \leq i' \leq N-1$ such that $N \nmid (a_{k,0} - i')$ for $k \geq 0$. Clearly $L \cup \{i'\}$ is a D-set in Q , which contradicts the maximality of L . This proves the claim. We rewrite (4.7) as

$$\{a_{k,0} : k \geq 0\} = \{i_0 + N\alpha_{i_0,0} : i_0 \in \Omega_N\} \subseteq C.$$

From the claim, we can define ι on Ω_N by $\iota(i) = i + N\alpha_{i,0}$, $i = 0, 1, \dots, N-1$ and in particular $\iota(0) = 0$. Similarly we can show that, for each $0 \leq i_0 \leq N-1$,

$$\{a_{k,1} : a_{k,0} = i_0 + N\alpha_{i_0,0}\}_{k=0}^\infty = \{i_1 + N\alpha_{i_1,1} : i_1 \in \Omega_N\} \subseteq C$$

and define $\iota(i_0 i) = i + N\alpha_{i,1}$, $i = 0, 1, \dots, N-1$. Again, we can show that

$$\{a_{k,2} : a_{k,0} = i_0 + N\alpha_{i_0,0} \text{ and } a_{k,1} = i_1 + N\alpha_{i_1,1}\}_{k=0}^{\infty} = \{i_2 + N\alpha_{i_2,2} : i_2 \in \Omega_N\}$$

and define $\iota(i_0 i_1 i) = i + N\alpha_{i,2}$, $i = 0, 1, \dots, N-1$. Inductively, we can define a map ι on Ω_N^* (with $\iota(\emptyset) = 0$). By the construction of ι , it is easy to see that (i) and (ii) in Definition 4.3 are satisfied. For any $\mathbf{i} = i_0 i_1 \dots i_n \in \Omega_N^*$ with $i_n \neq 0$, again by the construction of ι , there exists infinitely many λ_k such that $a_{k,t} = i_t + N\alpha_{i_t,t}$ for $0 \leq t \leq n$. Fix such a k , if $k \geq l_k$, we have $\lambda_k = \sum_{n=0}^{\infty} a_{k,n} q^n = \iota^*(\mathbf{i})$; If $k < l_k$, there exists $\mathbf{j} = j_{n+1} j_{n+2} \dots j_{l_k}$ such that $a_{k,t} = \iota(i_0 \dots i_n j_{n+1} \dots j_t)$ for $n+1 \leq t \leq l_k$. Then

$$\lambda_k = \sum_{n=0}^{\infty} a_{k,n} q^n = \iota^*(\mathbf{ij}).$$

This implies that (iii) in Definition 4.3 holds. Hence, ι is a selection mapping and $L \subseteq \iota^*(\Omega_N^t)$. The necessity follows by the maximal property of L and the proof of the sufficiency. \square

It follows directly from Corollary 4.2 and Proposition 4.4 that

Theorem 4.5. *Suppose $\rho = p/q$ and $N \mid q$. Then $\Lambda \subset \mathcal{Z}(\widehat{\mu}_{\rho,N})$ is a maximal bi-zero set if and only if there exist $m_0 \geq 1$ and a selection map ι such that $\Lambda = \rho^{-m_0} N^{-1}(\iota^*(\Omega_N^t))$.*

In particular, we see that for $p = 1$, the spectrum Λ in (2.1) corresponding to the case $m_0 = 1$ and the selection map ι is to take $\iota(\mathbf{i}) = i_k$ in (4.5). Also we observe that $\iota^*(\Omega_N^t)$ is an infinite set, we have

Corollary 4.6. *Suppose $\rho = p/q$ and $N \mid q$, then $L^2(\mu_{\rho,N})$ admits an infinite exponential orthonormal set.*

5. Spectrality for rational contraction

In this section, we prove the necessity of Theorem 1.1 when ρ is a rational number.

Proposition 5.1. *Let $\rho = \frac{p}{q}$ and $1 \leq \gcd(N, q) < N$, then $\mu_{\rho,N}$ is not a spectral measure.*

Proof. Suppose on the contrary that $\mu_{\rho,N}$ is a spectral measure. Let Λ be a spectrum of $\mu_{\rho,N}$ with $0 \in \Lambda$. Denote $d = \gcd(N, q)$. If $d = 1$, by Lemma 4.1, we have

$$\Lambda \subseteq p^{-m_0} \{q^m \frac{a}{N} : m \geq m_0, a \in \mathbb{Z} \setminus q\mathbb{Z}\} \cup \{0\}.$$

Denote $\mathcal{D} = \{0, 1, \dots, N-1\}$ and let $\mu' = \delta_{\rho^1 \mathcal{D}} * \delta_{\rho^2 \mathcal{D}} * \dots * \delta_{\rho^{m_0} \mathcal{D}} * \delta_{\rho^{m_0+2} \mathcal{D}} * \dots$ be the convolution of $\delta_{\rho^k \mathcal{D}}$ for $k \geq 1$ and $k \neq m_0 + 1$ (here $\delta_A = \frac{1}{\#A} \sum_{a \in A} \delta_a$ and δ_a is the Dirac measure). Then $\mu_{\rho,N} = \delta_{\rho^{m_0+1} \mathcal{D}} * \mu'$. We claim that Λ is a bi-zero set of μ' . The claim leads to a contradiction by Lemma 2.2. We prove the claim by assuming that $\rho^{-m_0-1} \frac{a}{N} \in \Lambda - \Lambda$ where $a \in \mathbb{Z} \setminus N\mathbb{Z}$. Then there exist k, l such that

$$\rho^{-m_0-1} \frac{a}{N} = p^{-m_0} q^{-m_k} \frac{a_k}{N} - p^{-m_0} q^{-m_l} \frac{a_l}{N},$$

where $a_k, a_l \in (\mathbb{Z} \setminus q\mathbb{Z}) \cup \{0\}$. Then $p \mid a$. Hence $\rho^{-m_0-1} \frac{a}{N} = \rho^{-m_0} \frac{qa/p}{N} \in \mathcal{Z}(M_{\rho^{m_0}N})$ and the claim follows; If $1 < d < N$, write $N = N'd, q = q'd$. Then $\mathcal{D} = C + d\mathcal{E}$, where $C = \{0, 1, \dots, d-1\}$ and $\mathcal{E} = \{0, 1, \dots, N'-1\}$. Note that $M_N(\xi) = M_d(\xi)M_{N'}(d\xi)$ and

$$\widehat{\mu}_{\rho, N}(\xi) = \prod_{k=1}^{\infty} M_N(\rho^k \xi) = \prod_{k=1}^{\infty} M_d(\rho^k \xi) \prod_{k=1}^{\infty} M_{N'}(\rho^k d\xi).$$

Let ν be the probability measure such that

$$\widehat{\nu}(\xi) = \prod_{k=1}^{\infty} M_d(\rho^k \xi) \prod_{k \geq 1, k \neq m_0+1} M_{N'}(\rho^k d\xi).$$

Then $\mu = \nu * \delta_{\rho^{m_0+1}d\mathcal{E}}$. We claim that Λ is a bi-zero set of ν . Hence the proposition follows by Lemma 2.2 again.

To prove the claim, we let $\eta \in (\Lambda - \Lambda) \setminus \{0\} (\subset \mathcal{Z}(\mu_{\rho, N}))$, then either $\eta \in \mathcal{Z}(\nu)$ or $\eta \in \mathcal{Z}(M_{N'}(\rho^{m_0+1}d(\cdot)))$. The first case satisfies the claim trivially. Hence we need only consider the second case, i.e., there exists $\eta \in (\Lambda - \Lambda)$ such that $\eta \in \mathcal{Z}(M_{N'}(\rho^{m_0+1}d(\cdot)))$. By (2.3), we have $\eta = \frac{1}{\rho^{m_0+1}d} \frac{a}{N'} (= (\frac{q}{p})^{m_0+1} \frac{a}{N})$ with $N' \nmid a$; also by (4.2), there exist k, ℓ such that

$$\eta = p^{-m_0} q^{m_k} \frac{c_k}{N} - p^{-m_0} q^{m_\ell} \frac{c_\ell}{N},$$

where $q \nmid c_k$ and $q \nmid c_\ell$. Hence we have $\frac{q^{m_0+1}a}{p} = q^{m_k} c_k - q^{m_\ell} c_\ell$. This implies that $p \mid a$. By letting $a' = qa/p$, we see that $N \nmid a'$ (as $N' \nmid a$). Therefore

$$\eta = (\frac{q}{p})^{m_0+1} \frac{a}{N} = (\frac{q}{p})^{m_0} \frac{a'}{N} \in \mathcal{Z}(M_N(\rho^{m_0}(\cdot))).$$

As $\mathcal{Z}(M_N(\rho^{m_0}(\cdot))) \subset \mathcal{Z}(\nu)$, the claim follows. \square

Proposition 5.2. *Let $\rho = \frac{p}{q}$. If $q \mid N$ and $p > 1$, then $\mu_{\rho, N}$ is not a spectral measure.*

The proof of this case is more elaborate. We show that any maximal bi-zero set Λ of $\mu_{\rho, N}$ does not satisfy the condition on $Q(\xi)$ in Theorem 2.1(ii). To this end, we define

$$\mu_n = \delta_{\rho\Omega_N} * \dots * \delta_{\rho^n\Omega_N}$$

for $n \geq 1$. Then

$$\widehat{\mu}_n(\xi) = \prod_{i=1}^n M_N(\rho^i \xi) \quad \text{and} \quad \widehat{\mu}_{\rho, N}(\xi) = \widehat{\mu}_n(\xi) \widehat{\mu}_{\rho, N}(\rho^n \xi).$$

We need a few technical lemmas.

Lemma 5.3. *Let ι be a selection mapping. Then*

$$\sum_{\mathbf{i} \in \Omega_N^n, |\mathbf{i}| \leq n} |\widehat{\mu}_{n+m_0-1}(\xi + \rho^{-m_0} N^{-1} \iota^*(\mathbf{i}))|^2 \leq 1$$

for $n \geq 1$ and $\xi \in \mathbb{R}$.

Proof. First we prove the case for $m_0 = 1$. According to the Bessel inequality, it suffices to show that $\rho^{-1}N^{-1}\iota^*(\{\mathbf{i} \in \Sigma_N^\iota : |\mathbf{i}| \leq n\})$ is a bi-zero set of μ_n . For any $\mathbf{i}, \mathbf{j} \in \Sigma_N^\iota$, $\mathbf{i} \neq \mathbf{j}$ and $1 \leq |\mathbf{i}|, |\mathbf{j}| \leq n$, we let $\mathbf{i}' = \mathbf{i}0^{n-|\mathbf{i}|} := i'_1 \cdots i'_n$ and $\mathbf{j}' = \mathbf{j}0^{n-|\mathbf{j}|} := j'_1 \cdots j'_n$. Let s be the smallest integer such that $i'_s \neq j'_s$. Then $s \leq n$ and

$$\iota^*(\mathbf{i}) - \iota^*(\mathbf{j}) = (\iota(\mathbf{i}'|_s) - \iota(\mathbf{j}'|_s))q^{s-1} + \alpha q^s$$

for some integer α . By (4.5), $\iota(\mathbf{i}'|_s) - \iota(\mathbf{j}'|_s)$ is not divisible by N . It follows from (2.2) that,

$$M_N(\rho^s \rho^{-1} N^{-1} (\iota^*(\mathbf{i}) - \iota^*(\mathbf{j}))) = M_N\left(\frac{p^{s-1}(\iota(\mathbf{i}'|_s) - \iota(\mathbf{j}'|_s))}{N}\right) = 0.$$

This implies that $\widehat{\mu}_n(\rho^{-1}N^{-1}(\iota^*(\mathbf{i}) - \iota^*(\mathbf{j}))) = 0$. Similarly, we have $\widehat{\mu}_n(\rho^{-1}N^{-1}\iota^*(\mathbf{i})) = 0$ for any $\mathbf{i} \in \Sigma_N^\iota$ and $0 < |\mathbf{i}| \leq n$. By Theorem 2.1,

$$\sum_{\mathbf{i} \in \Omega_N^\iota, |\mathbf{i}| \leq n} |\widehat{\mu}_n(\xi + \rho^{-1}N^{-1}\iota^*(\mathbf{i}))|^2 \leq 1$$

This completes the proof for $m_0 = 1$. For $m_0 > 1$, we observe that

$$|\widehat{\mu}_{n+m_0-1}(\xi)| = |\widehat{\mu}_{m_0-1}(\xi)| |\widehat{\mu}_n(\rho^{m_0-1}\xi)| \leq |\widehat{\mu}_n(\rho^{m_0-1}\xi)|$$

and apply the inequality. The result follows. \square

The following lemma is a simple generalization of Lemma 2.10 in [4].

Lemma 5.4. *Let $a = \ln p / \ln q$. Then for any $\xi > 1$ there exists ξ' such that $\rho^2 \xi^a \leq \xi' \leq \rho \xi$ and*

$$|\widehat{\mu}_{\rho, N}(\xi)| \leq c |\widehat{\mu}_{\rho, N}(\xi')|,$$

where $c = \max \{ |M_N(\xi)| : \frac{1}{2q} \leq \xi \leq \frac{1}{2} \} < 1$.

Proof. For any $x \in \mathbb{R}$, denote the unique number $\langle x \rangle$ that satisfies $\langle x \rangle \in (-1/2, 1/2]$ and $x - \langle x \rangle \in \mathbb{Z}$. If $\langle \rho \xi \rangle \notin (-\frac{1}{2q}, \frac{1}{2q})$, then

$$|\widehat{\mu}_{\rho, N}(\xi)| = |M_N(\rho \xi)| |\widehat{\mu}_{\rho, N}(\rho \xi)| = |M_N(\langle \rho \xi \rangle)| |\widehat{\mu}_{\rho, N}(\rho \xi)| \leq c |\widehat{\mu}_{\rho, N}(\rho \xi)|.$$

Hence we obtain the desired inequality by letting $\xi' = \rho \xi$; If $\langle \rho \xi \rangle \in (-\frac{1}{2q}, \frac{1}{2q})$, then

$$(5.1) \quad \rho \xi - \langle \rho \xi \rangle = r_t q^t + \cdots + r_l q^l,$$

where $0 \leq r_j < q$ for $t \leq j \leq l$ and $r_l > 0$. Then

$$\langle \rho^{t+2} \xi \rangle = \langle \rho^{t+1} \langle \rho \xi \rangle + \frac{r_t p^{t+1}}{q} \rangle.$$

Note that $|\rho^{t+1} \langle \rho \xi \rangle| < \frac{1}{2q}$ and $\frac{1}{q} \leq |\langle \frac{r_t p^{t+1}}{q} \rangle| \leq \frac{q-1}{q}$, then $\langle \rho^{t+2} \xi \rangle \notin (-\frac{1}{2q}, \frac{1}{2q})$.

By (5.1), we have $\xi \geq q^t$, which implies $\rho^t \geq \xi^{a-1}$, where $a = \ln p / \ln q$. Let $\xi' = \rho^{t+2} \xi$, then $\xi' \geq \rho^2 \xi^a$, and hence

$$\begin{aligned} |\widehat{\mu}(\xi)| &= |M_N(\rho \xi)| \cdots |M_N(\rho^{t+2} \xi)| |\widehat{\mu}(\rho^{t+2} \xi)| \\ &\leq |M_N(\langle \rho^{t+2} \xi \rangle)| |\widehat{\mu}(\rho^{t+2} \xi)| \\ &\leq c |\widehat{\mu}(\xi')|. \end{aligned}$$

\square

Lemma 5.5. Assume $p > 1$, then there exist integers $b \geq 2$, $n_0 \geq 2$, and real number $\beta > 1$, $C > 1$ such that for any $\mathbf{i} \in \Omega_N^\ell$ with $n^b < |\mathbf{i}| \leq (n+1)^b$, $n \geq n_0$, we have

$$\left| \widehat{\mu}_{\rho, N} \left(\rho^{(n+1)^b + (m_0-1)} (\xi + \rho^{-m_0} N^{-1} \iota^*(\mathbf{i})) \right) \right| \leq \frac{C}{n^\beta}$$

for $0 \leq \xi \leq \frac{1}{2\rho^{m_0}N}$.

Proof. Note that $p > 1$ implies that $q > 2$. Let b be an integer such that $b > 1 + \frac{\log a}{\log c}$, where $a = \log p / \log q$, and c is as in Lemma 5.4. Since $\iota^*(\mathbf{i}) = \sum_{j=1}^\infty \iota(\mathbf{i} 0^\infty|_j) q^{j-1}$ for any $\mathbf{i} \in \Sigma^\ell$. Let ℓ be the largest index such that $\iota(\mathbf{i} 0^\infty|_\ell) \neq 0$. Then $\ell \geq n^b + 1$, and a direct estimation shows that

$$|\iota^*(\mathbf{i})| \geq q^{\ell-1} - (q-2) \sum_{j=1}^{\ell-1} q^{j-1} \geq q^{\ell-3} + \frac{1}{2}.$$

This together with the assumption on ξ implies that

$$\left| \rho^{m_0-1} \xi + \frac{\iota^*(\mathbf{i})}{\rho N} \right| \geq \frac{|\iota^*(\mathbf{i})|}{\rho N} - \frac{1}{2\rho N} \geq q^{\ell-4} \geq q^{n^b-3}.$$

Let $\eta = \rho^{(n+1)^b} (\rho^{m_0-1} \xi + \frac{\iota^*(\mathbf{i})}{\rho N})$. It is easy to see that if n large enough, then $(n+1)^b + 3 \leq n^b + b^2 n^{b-1}$. Hence if we take a large n_0 , then for $n \geq n_0$,

$$|\eta| \geq \frac{p^{(n+1)^b}}{q^{(n+1)^b - n^b + 3}} \geq \frac{p^{(n+1)^b}}{q^{b^2 n^{b-1}}} = \left(\frac{p^{(1+1/n)^b n}}{q^{b^2}} \right)^{n^{b-1}} \geq \left(\frac{p^n}{q^{b^2}} \right)^{n^{b-1}} \geq q^{n^{b-1}}.$$

Applying Lemma 5.4 to $\eta_i = \rho^i \eta$ recursively, we have

$$|\widehat{\mu}(\eta)| \leq c |\widehat{\mu}(\eta_1)| \leq \cdots \leq c^l |\widehat{\mu}(\eta_l)|$$

as long as $|\eta_l| \geq 1$. This is the case if we let $l = \lceil \log_a \frac{2n^{1-b}}{1-a} \rceil$, because

$$|\eta_l| \geq \rho^2 |\eta_{l-1}|^a \geq \cdots \geq \rho^{2+2a+\cdots+2a^{l-1}} |\eta|^{a^l} \geq \rho^{\frac{2}{1-a}} |\eta|^{a^l} > q^{-\frac{2}{1-a}} q^{n^{b-1} a^l} \geq 1.$$

Hence,

$$|\widehat{\mu}(\eta)| \leq c^l \leq c^{\log_a \frac{2n^{1-b}}{1-a}} = c^{\log_a \frac{2}{1-a}} n^{-(b-1) \log c / \log a}.$$

The lemma follows by assigning C and β in the obvious way. \square

Proof of Proposition 5.2. We assume all the parameters in Lemma 5.5. To simplify the notations, we write $\mu = \mu_{\rho, N}$, $\alpha(\mathbf{i}) = \rho^{-m_0} N^{-1} \iota^*(\mathbf{i})$, $\mathcal{I}_n = \{\mathbf{i} \in \Omega_N^\ell, |\mathbf{i}| \leq n^b\}$, $\mathcal{I}_{n, n+1} = \{\mathbf{i} \in \Omega_N^\ell, n^b < |\mathbf{i}| \leq (n+1)^b\}$. Let

$$Q_n(\xi) = \sum_{\mathbf{i} \in \mathcal{I}_n} |\widehat{\mu}(\xi + \alpha(\mathbf{i}))|^2.$$

Then

$$\begin{aligned}
Q_{n+1}(\xi) &= Q_n(\xi) + \sum_{I_{n,n+1}} |\widehat{\mu}(\xi + \alpha(\mathbf{i}))|^2 \\
&= Q_n(\xi) + \sum_{I_{n,n+1}} |\widehat{\mu}_{(n+1)^b+m_0-1}(\xi + \alpha(\mathbf{i}))|^2 |\widehat{\mu}(\rho^{(n+1)^b+m_0-1}(\xi + \alpha(\mathbf{i})))|^2 \\
&\leq Q_n(\xi) + \frac{C^2}{n^{2\beta}} \sum_{I_{n,n+1}} |\widehat{\mu}_{(n+1)^b+m_0-1}(\xi + \alpha(\mathbf{i}))|^2 \quad (\text{by Lemma 5.5}) \\
&\leq Q_n(\xi) + \frac{C^2}{n^{2\beta}} \left(1 - \sum_{I_n} |\widehat{\mu}_{(n+1)^b+m_0-1}(\xi + \alpha(\mathbf{i}))|^2\right) \quad (\text{by Lemma 5.4}) \\
&\leq Q_n(\xi) + \frac{C^2}{n^{2\beta}} (1 - Q_n(\xi)).
\end{aligned}$$

This implies that $n > n_0$,

$$1 - Q_{n+1}(\xi) \geq \left(1 - Q_n(\xi)\right) \left(1 - \frac{C^2}{n^{2\beta}}\right) \geq \cdots \geq (1 - Q_{n_0}(\xi)) \prod_{k=n_0}^n \left(1 - \frac{C^2}{k^{2\beta}}\right).$$

Now let $Q(\xi) = \sum_{\mathbf{i} \in \Omega'_N} |\widehat{\mu}(\xi + \alpha(\mathbf{i}))|^2$, it is the sum over a maximal bi-zero set (by Proposition 4.5). The above implies

$$1 - Q(\xi) \geq C' (1 - Q_{n_0}(\xi)).$$

where $C' = \prod_{k=n_0}^{\infty} (1 - \frac{C^2}{k^{2\beta}}) \neq 0$. This implies that $Q(\xi) \neq 1$, and hence by Theorem 2.1 and Proposition 4.5, any maximal bi-zero set of $\mu_{p,N}$ cannot be a spectrum when $\rho = p/q$, p, q co-prime, and $p \neq 1$. \square

6. Remarks

It was proved in [13] that if μ is a spectral self-similar measure with support in $[0, 1]$ and ν is a probability counting measure support on a finite set in \mathbb{Z} , then the convolution $\mu * \nu$ is a spectral measure if and only if ν is a spectral measure. It was pointed out by Gabardo and Lai (private communication) that if both μ and ν are two probability measures with $\mu * \nu = L|_{[0,1]}$, where $L|_{[0,1]}$ is the Lebesgue measure restricted on $[0, 1]$, then both μ and ν are spectral measures (which is a corollary of the main results in [1] and [21]). It has been asked:

Is the convolution of two spectral self-similar measures with essentially disjoint supports a spectral measure?

The question can be answered by Theorem 1.1. Observe that $\{0, 1, 2, 3\} = \{0, 1\} \oplus \{0, 2\}$, hence

$$\mu_{1/6, 4} = \mu_{1/6, 2} * \mu_{1/6, \{0,2\}}.$$

It follows that both $\mu_{1/6, 2}$ and $\mu_{1/6, \{0,2\}}$ are spectral measures (by [15] or Theorem 1.1), but Theorem 1.1 implies that $\mu_{1/6,4}$ is not a spectral measure. As a consequence, convolution of two spectral measures may not be spectral.

One of the challenge questions on the spectral measures is the conjecture of Łaba and Wang [18]:

Let μ be a self-similar measure as in (1.1), then μ is a spectral measure if and only if (i) $w_j = 1/N$; (ii) $\rho = 1/q$ for some integer $q > 1$; and (iii) there exist a constant c and an integer digit set \mathcal{D}' such that $\mathcal{D} = c\mathcal{D}'$ and $\mathcal{D}' \oplus \mathcal{B} \equiv \{0, \dots, q-1\} \pmod{q}$ for some $\mathcal{B} \subset \mathbb{Z}$.

In [10], it was shown that (i) is necessary for a spectral measure under the no overlap condition. Our Theorem 1.1 settles the case where $\mathcal{D} = \{0, \dots, N-1\}$. The digit set \mathcal{D}' in (iii) is called an *integer tile*. The study of integer tiles has a long history related to the geometry of numbers ([2] and the references there), and the spectral property of \mathcal{D} as a discrete set itself is still unsolved [21].

As was proved in [15], the Cantor measure $\mu_{1/k}$ with k an odd integer is not a spectral measure. It is well known that a relaxing of the orthonormal basis is the concept of *frame* introduced by Duffin and Schaeffer in the 50's (see [3]). We call a measure μ an *F-spectral measure* (F for frame) if there exists a countable set $\{e_\lambda : \lambda \in \Lambda\}$ and $A, B > 0$ such that for any $f \in L^2(\mu)$,

$$A\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, e_\lambda \rangle|^2 \leq B\|f\|^2,$$

and call μ a *R-spectral measure* if in addition it is a basis (R for Riesz). The frame structure of $L^2[0, 1]$ has been studied in detail in [20, 24]; also there are extensive studies of the frames on $L^2(\mu)$ [5, 25]. However the basic problem whether $\mu_{1/k}$ with k an odd integer, in particular for $\mu_{1/3}$, is an F-(or R-)spectral measure is still unresolved.

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